Solutions for EIT circuits review

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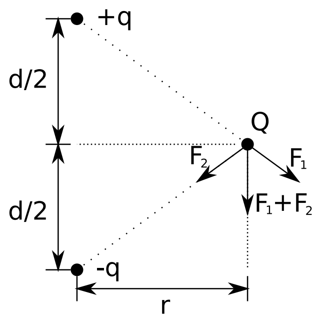
Basics

1. Charges q and $-q$ are separated by distance d and charge Q is placed at distance r from the midpoint, as shown in the figure below. Find the net force on the charge Q by the other charges.

   Answer: The forces due to charges $+q$ and $-q$ add as shown in the figure below,



   As we can see, the horizontal components of the two forces cancel out, so we only worry about the vertical component.

   The vertical component of the force due to one of two charges is,

$$F_y = \left(\frac{qQ}{4\pi\epsilon_0 R^2}\hat{R}\right)\cdot(-\hat{y})$$
$$= \left(\frac{qQ}{4\pi\epsilon_0[r^2+(d/2)^2]}\right)\left(-\frac{d/2}{\sqrt{r^2+(d/2)^2}}\right)$$
$$= -\frac{qQd}{8\pi\epsilon_0(r^2+d^2/4)^{3/2}},$$

   so the total force on the charge Q is,

$$\vec{F} = -\frac{qQd}{4\pi\epsilon_0(r^2+d^2/4)^{3/2}}\hat{y}.$$

2. Two plates of area A, each carrying charges of $+Q$ and $-Q$ are d distance apart. How much work needs to be done to mave a charge q ($q > 0$) from the negatively charged plate to the positively charged plate?

   Answer: We approximate the electric field between the two plates by the electric field due to charged infinite planes (this approximation is valid if A is much larger than d^2).

   Then, the electric field due to a single infinite plate with charge density of $\sigma = Q/A$ is,

$$\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{n} = \frac{Q/A}{2\epsilon_0}\hat{n},$$

   where $\hat{n}$ is the normal vector pointing away from the plate.

   In the region between the two plates, the electric fields add (they are in the same direction), and in the region outside, they cancel (they are in the opposite direction; same magnitude), so,

$$\vec{E}_{\text{tot}} = 2\frac{Q/A}{2\epsilon_0} = \frac{Q}{\epsilon_0 A}.$$

   The work done can be calculated by calculating the voltage difference between the two plates and using $\Delta PE = q\Delta V$. The voltage difference is,

$$\Delta V_{ab} = \int_a^b \vec{E}\cdot d\vec{l},$$

   but the electric field is constant and parallel to displacement, so the integral is simply the electric field times the distance,

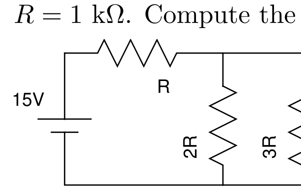
$$= EL = \frac{Q}{\epsilon_0 A}\cdot d = \frac{Qd}{\epsilon_0 A},$$

   so the work done is,

$$W = q\Delta V_{ab} = \frac{qQd}{\epsilon_0 A}.$$

DC circuits

3. $R = 1$ kΩ. Compute the following:



(a) the voltage change across the resistor R, and
(b) the power output of the battery.

Answer:

(a) The voltage drop across a resistor is given by,

$$\Delta V = IR.$$

The current (I) through the resistor R in this circuit is equal to the current from the battery itself, so we can find find I by simplifying the circuit until it can be represented by the single battery and a single Thevenin equivalent resistor.

First we add $2R$ and $3R$ in parallel,

$$\frac{1}{R_{\text{eq}}} = \frac{1}{2R} + \frac{1}{3R},$$

so,

$$R_{\text{eq}} = \frac{6}{5}R.$$

Then, we add this with R in series to obtain the total resistance,

$$R_{\text{tot}} = R + \frac{6}{5}R = \frac{11}{5}R,$$

and the current out of the battery is given by Ohm's law,

$$I = \frac{V}{R_{\text{tot}}} = \frac{V}{11R/5} = \frac{5}{11}\frac{V}{R},$$

and the voltage drop across R is,

$$\Delta V = \left(\frac{5}{11}\frac{V}{R}\right)R = \frac{5}{11}V \approx 6.8 \text{ V}.$$

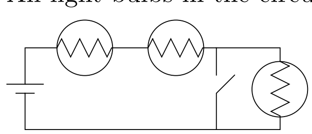
(b) The power output of the battery is given by,

$$P = VI.$$

We calculated current I out of battery in part (a), so the power output is,

$$P = V \cdot \frac{5}{11}\frac{V}{R} = \frac{5}{11}\frac{V^2}{R} \approx 0.102 \text{ W}.$$

4. All light bulbs in the circuit below are identical.



What happens to the following quantities when the switch is closed? (a) current through the battery, (b) brightness of each bulbs, (c) the voltage drop across each bulbs, and (d) total power dissipated by the bulbs.

Answer:

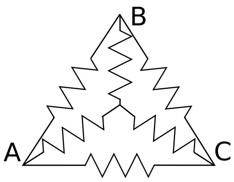
(a) Total resistance in the circuit decreases when one of the light bulb is shorted (imagine adding a very small resistor to r to a large resistor R in parallel—the resulting equivalent resistance is less than r itself). So current output increases.

(b) With more current, the light bulbs through which this current passes use more energy, so they increase in brightness. However, there is no current through the one light bulb that was short-circuited, so the light bulb to the far right is off.

(c) The voltage drop across the two light bulbs at the top increases, as $\Delta V = IR$ and current increased. But the voltage drop across the light bulb on the right is zero, because no current goes through it.

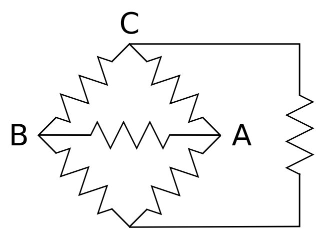
(d) The total power dissipated by the light bulbs is the same power expended by the battery, which is given by $P = VI$, and since the current increased while the voltage output of the battery stayed the same, the total power output increases.

5. All resistors are 1 kΩ.



Find the Thevenin equivalent resistance between the points A and B; and B and C.

Answer: This problem can be simplified using symmetry. For consideration of resistance between A and B, the symmetry of the resistor network becomes more apparent if we rearrange the circuit into the below shape:



If we put a voltage difference of, for example, $\Delta V =$ 1 V between terminals A and B, by symmetry, there can be no current through the resistor on the right. You can argue this two ways: (1) if you assume that there is a current in a particular direction and reflect the whole circuit across a line connecting A and B, then you get the same circuit back, except with the direction of the current reversed. The only way these two pictures can be both right is if current is zero. (2) From the symmetry, the voltage at point C and at point across from C must be the same, so there is no voltage difference across the resistor on the right.

So, since there is no current through the resistor on the right, we can ignore it in further analysis of the circuit, and we can simplify the remaining resistors. The two sets of resistors connecting A and B either A-C-B, or on the opposite side are two resistors in series, so we can replace each with a single resistor of resistance $2R$.

Then, we have three resistors, $2R$, R, and $2R$ connecting A and B in parallel, so adding them in parallel,

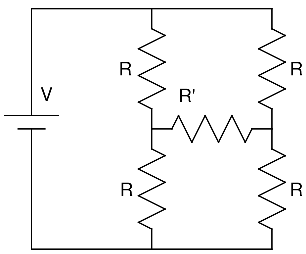
$$\frac{1}{R_{\text{eq}}} = \frac{1}{2R} + \frac{1}{R} + \frac{1}{2R} = \frac{2}{R},$$

so,

$$R_{\text{eq}} = \frac{R}{2}.$$

As for resistance between B and C, it is the same as resistance between A and B by symmetry (imagine rotating the original circuit by 60°).

6. Consider the circuit shown below. V and R are given quantities.

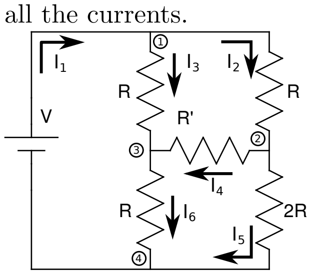


(a) Find the current through the resistor R'.
(b) One of the R resistors is changed to a $2R$ resistor. What is the current through R'?

Answer:

(a) By symmetry, the current through R' is zero. Here is one detailed argument: assuming that there is a current I through R', we have to pick a direction for the current. But when you reflect the whole circuit along the vertical line that bisects the resistor network horizontally, we get the exact same resistor network back, except with the direction of current reversed. The only way to avoid a contradiction is to say $I = 0$. Similar symmetry argument can be made using voltages at the two ends of R' as well.

(b) Changing one of the resistors into $2R$ breaks the symmetry, and we have to do a full-blown calculation using Kirchhoff's rules to find the current through R'. Below is the circuit diagram with the new resistor value and label for all the currents.



The goal is to solve for our 6 unknowns (six values of current) using 6 equations. We can use Kirchhoff's junction rule and loop rule to come up with the necessary 6 equations.

As labeled, we have 4 junctions, we can use 3 out of 4 (the last junction will always give an equation that is not independent from other equations), from junctions 1, 2, and 4:

$$\begin{aligned} 0 &= I_1 - I_2 - I_3 \\ 0 &= I_2 - I_4 - I_5 \\ 0 &= I_6 + I_5 - I_1. \end{aligned}$$

We still need 3 more equations, and for those, we use the loop rule. We choose the loops as small as possible while avoiding overlap as much as possible. The first loop can be made with the top right rectangle (keep in mind that voltage change is negative when crossing a resistor along with the current, and the voltage change is positive when crossing a resistor against the current),

$$-I_2R - I_4R' + I_3R = 0,$$

and the second loop from the bottom right rectangle,

$$-I_5(2R) + I_6R + I_4R' = 0,$$

and the third loop from the left rectangle (the sign of voltage change when going across the battery is given by the polarity of battery: voltage rise if going from negative terminal to positive terminal; voltage drop otherwise),

$$V - I_3R - I_6R = 0.$$

And we meticuluously eliminate all variables but I_4. (If you have access to a computer algebra system that can invert matrices, then building up a coefficient matrix with the above equations may be an easier way.)

First eliminating I_1, we obtain the following 5 equations,

$$\begin{aligned} 0 &= I_2 - I_4 - I_5 \\ 0 &= I_6 + I_5 - I_2 - I_3 \\ 0 &= -I_2R - I_4R' + I_3R \\ 0 &= -2I_5R + I_6R + I_4R' \\ 0 &= V - I_3R - I_6R, \end{aligned}$$

then we eliminate I_2,

$$\begin{aligned} 0 &= I_6 + I_5 - I_4 - I_5 - I_3 = I_6 - I_4 - I_3 \\ 0 &= -(I_4 + I_5)R - I_4R' + I_3R \\ 0 &= -2I_5R + I_6R + I_4R' \\ 0 &= V - I_3R - I_6R, \end{aligned}$$

then we eliminate I_3,

$$\begin{aligned} 0 &= -(I_4 + I_5)R - I_4R' + (I_6 - I_4)R \\ &= -2I_4R - I_5R + I_6R - I_4R' \\ 0 &= -2I_5R + I_6R + I_4R' \\ 0 &= V - (I_6 - I_4)R - I_6R = V - 2I_6R + I_4R, \end{aligned}$$

then we eliminate I_5 using the second equation above,

$$I_5 = \frac{I_6}{2} + \frac{I_4 R'}{2R},$$

$$\begin{aligned} 0 &= -2I_4R - \left(\frac{I_6}{2} + \frac{I_4R'}{2R}\right)R + I_6R - I_4R' \\ &= -2I_4R + 0.5I_6R - 1.5I_4R' \\ 0 &= V - 2I_6R + I_4R, \end{aligned}$$

and finally, using the second equation, we eliminate I_6,

$$I_6 = 0.5V/R + 0.5I_4,$$

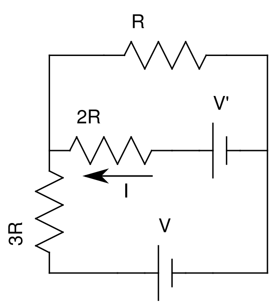
which leaves us with,

$$\begin{aligned} 0 &= -2I_4R + 0.5(0.5V/R + 0.5I_4)R - 1.5I_4R' \\ &= I_4\left(-2R + \frac{1}{4}R - \frac{3}{2}R'\right) + \frac{1}{4}V \end{aligned}$$

and we solve it for I_4:

$$I_4 = \frac{1}{4}\frac{V}{7R/4 + 3R'/2} = \frac{V}{7R + 6R'}.$$

7. Consider the circuit shown below.



Given the current I through the $2R$ resistor in the direction indicated, find V' in terms of V and R.

Answer: For this problem, it is necessary to use Kirchhoff's rules, as it is not possible to simplify the resistors to obtain a one-battery, one-resistor circuit. First, we identify all the unknowns (i.e. currents and component values that we do not know): V', I_1 (current through R), and I_2 (current through $3R$ and V).

So, we need 3 equations, and since there are two junctions, we can only use one of them to get one equation:

$$I - I_1 - I_2 = 0,$$

and we use the loop rule to get the remaining two equations. We use the bottom loop and the top loop to get,

$$\begin{aligned} 0 &= V' - I(2R) - I_2(3R) - V \\ 0 &= V' - I(2R) - I_1R; \end{aligned}$$

in the first equation, V term gets a negative sign because as we move along the loop counterclockwise, we are going across the battery V from positive terminal to negative terminal.

Now we solve for V' by carefully eliminating I_1 and I_2 from the equations. First, eliminating I_1, we obtain,

$$\begin{aligned} 0 &= V' - I(2R) - I_2(3R) - V \\ 0 &= V' - I(2R) - (I - I_2)R = V' - 3IR + I_2R, \end{aligned}$$

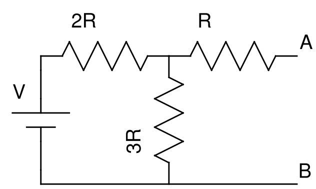
and we now eliminate I_2 by multiplying the bottom equation through by 3 and adding the two equations together:

$$0 = V' + 3V' - 2IR - 9IR - 3I_2R + 3I_2R - V = 4V' - 11IR - V,$$

and we solve for V':

$$V' = \frac{V + 11IR}{4}.$$

8. Consider the curcuit below. Find:

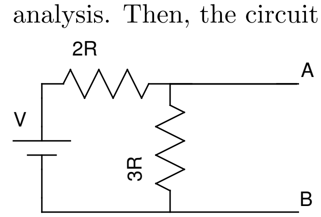


(a) the Thevenin equivalent voltage
(b) Thevenin equivalent resistance between points A and B

Answer:

(a) Thevenin equivalent voltage is the voltage between points A and B in "open circuit" (i.e. the voltage between the two points when they are not connected at all).

If A and B are connected, we see that the current through R is zero, so the voltage drop across R is zero, and we can ignore it in the analysis. Then, the circuit looks as below:



So, the voltage between A and B is the voltage drop across $3R$, which we calculate as follows. First, we find the current through the circuit, then the voltage drop across any resistor is given as the product of current and resistance. The total resistance in the circuit is given by adding the two resistors in series,

$$R_{\text{eq}} = 2R + 3R = 5R.$$

And the current is,

$$I = \frac{V}{5R}.$$

So the voltage drop across the $3R$ resistor is,

$$\Delta V = I(3R) = \frac{3}{5}V,$$

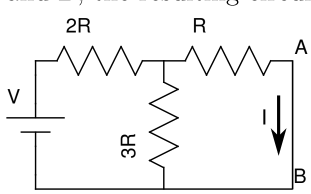
which is the Thevenin equivalent voltage of the circuit.

(b) There are two methods for finding the Thevenin equivalent resistance. The first method applies in the general case but can be somewhat lengthy; the second method applies in some special cases but is usually quicker when it works.

First method: First, short-circuit A and B (i.e. connect them with wires) and find current I through the wire connecting A and B. Then,

$$R_{\text{eq}} = \frac{V_{\text{eq}}}{I},$$

where V_{eq} is the Thevenin equivalent voltage found in part (a). When we short circuit A and B, the resulting circuit looks as below:



To find the current I, we find the voltage drop across R by simplifying the circuit. We can first replace $3R$ and R with a single resistor of resistance R_{3+1} by adding them in parallel:

$$\frac{1}{R_{3+1}} = \frac{1}{3R} + \frac{1}{R},$$

so,

$$R_{3+1} = \frac{3}{4}R,$$

and simplify the whole circuit down to one resistor by adding the remaing ones in series,

$$R_{3+1+2} = 2R + \frac{3}{4}R = \frac{11}{4}R,$$

then the current from battery is given by Ohm's law,

$$I = \frac{V}{R_{3+1+2}} = \frac{4}{11}\frac{V}{R},$$

and the voltage drop across the R_{3+1} resistor is,

$$\Delta V = IR_{3+1} = \frac{4}{11}\frac{3}{4}V = \frac{3}{11}V,$$

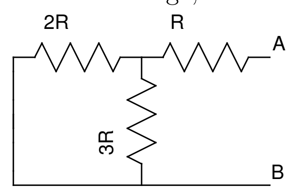
which is also the voltage drop across R. So the current through R is,

$$I = \frac{\Delta V}{R} = \frac{3}{11}\frac{V}{R}.$$

And as discussed above, the Thevenin equivalent resistance is given by,

$$R_{\text{eq}} = \frac{V_{\text{eq}}}{I} = \frac{(3/5)V}{(3/11)(V/R)} = \frac{11}{5}R.$$

Second method: We short-circuit all the batteries (i.e. replace them with wire) and find the resistance between points A and B with the resulting resistor network. After applying the prescribed change, the circuit looks like this:



To simplify this down to one equivalent resistor, we can first add $2R$ and $3R$ in parallel to get,

$$\frac{1}{R_{2+3}} = \frac{1}{2R} + \frac{1}{3R},$$

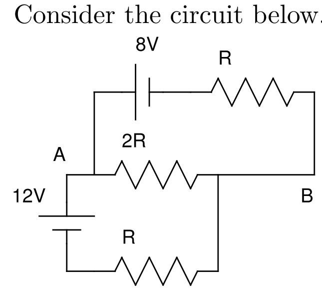
which gives,

$$R_{2+3} = \frac{6}{5}R.$$

And then we add this with R in series to get the Thevenin equivalent resistance:

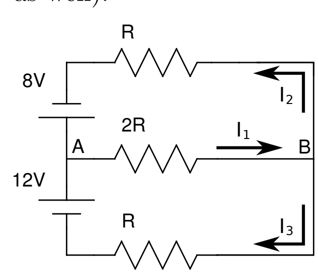
$$R_{\text{eq}} = \frac{6}{5}R + R = \frac{11}{5}R.$$

9. Consider the circuit below. $R = 1\ \text{k}\Omega$.



(a) Find the voltage difference between A and B;
(b) find the current through the resistor $2R$.

Answer: This is another application of Kirchhoff's rules. First, it might help to simplify the circuit diagram a little, as below (with labels for currents as well):



So, we have 3 unknowns, I_1, I_2, and I_3 (if we knew these three quantities, then we can answer any questions about the circuit in one single step), so we need three equations, which we get from Kirchhoff's rules.

First we use the junction rule. Since we have 2 junctions, we can use junction rule only once:

$$0 = I_1 - I_2 - I_3.$$

Then we use the loop rule to obtain the remaining 2 equations. We use the top loop and the bottom loop, going clockwise each time ($V_1 = 12$ V and $V_2 = 8$ V).

$$0 = V_1 - I_1(2R) - I_3R$$
$$0 = -V_2 + I_2R + I_1(2R).$$

To answer (a) and (b), we need to find I_1, so we eliminate I_2 and I_3 systematically. First, we eliminate I_2 to get,

$$0 = V_1 - 2I_1R - I_3R$$
$$0 = -V_2 + (I_1 - I_3)R + 2I_1R = -V_2 + 3I_1R - I_3R.$$

And we eliminate I_3 by subtracting the bottom equation from the top equation:

$$0 = V_1+V_2-2I_1R-3I_1R-I_3R+I_3R = V_1+V_2-5I_1R,$$

so, solving for I_1 (this answers (b)):

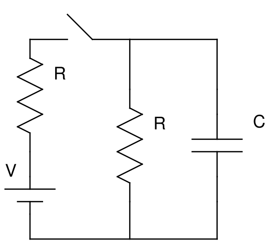
$$I_1 = \frac{V_1 + V_2}{5R} = \frac{20\text{ V}}{5(1\text{ k}\Omega)} = 4\text{ mA}.$$

And the voltage drop across $2R$ is,

$$\Delta V = I_1(2R) = \frac{V_1 + V_2}{5R}(2R) = \frac{2}{5}(V_1 + V_2) = \frac{2}{5}(20\text{ V}) = 8\text{ V}.$$

Capacitors and inductors

10. The switch is closed at time $t = 0$.



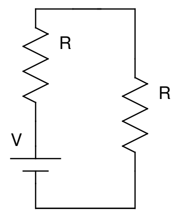
(a) What is the maximum charge on capacitor C at a very long time after $t = 0$?

(b) Once the capacitor is charged, switch is opened. Capacitor charge after $\Delta t = 2RC$?

Answer:

(a) To find the maximum charge on the capacitor, we need to find the voltage drop across the capacitor (then the charge is given by the definition of capacitance, $C = Q/\Delta V$).

When the capacitor is charged to maximum, the current through capacitor is zero, because by definition the charge on capacitor is not changing so no current is entering or leaving the capacitor. As we other circuit components, once we have determined the segment containing capacitor draws no current at all, we can remove it from the circuit in the analysis, and the circuit simplifies down to:



So the voltage across the second resistor R (which is also the voltage across the capacitor) is found by finding the current through the circuit,

$$I = \frac{V}{R+R} = \frac{V}{2R},$$

and multiplying it to the resistance:

$$\Delta V = IR = \frac{V}{2R}R = \frac{V}{2}.$$

So the amount of charge on the capacitor is,

$$Q = C\Delta V = \frac{V^2}{2}.$$

(b) Once the switch is opened, there is no current through the segment with battery and one resistor, so the circuit is a simple RC circuit with one capacitor discharging through the resistor R. You can solve the amount of charge $Q(t)$ as a function of time by using Kirchhoff's rule.

Using the loop rule (it's the only rule we can use for such a simple circuit),

$$\frac{Q(t)}{C} - IR = 0.$$

In order to solve for $Q(t)$, we need to relate the current I through the circuit to Q. This is the same current through the capacitor, and the current through the capacitor is related to the amount of charge on the capacitor by

$$I = -\frac{\mathrm{d}Q}{\mathrm{d}t},$$

because the current through a capacitor comes from the change in the amount of charge on the capacitor. Negative sign indicates that positive current coming out of capacitor corresponds to decrease in charge on capacitor.

So,

$$\frac{Q(t)}{C} + \frac{\mathrm{d}Q}{\mathrm{d}t}R = 0$$

This is a differential equation we can solve easily by separation of variables, which gives,

$$-\frac{\mathrm{d}t}{RC} = \frac{\mathrm{d}Q}{Q},$$

and integrating left and right from $t' = 0$ to $t' = t$,

$$-\frac{1}{RC}\int_0^t \mathrm{d}t = -\frac{t}{RC} = \int_0^t \frac{\mathrm{d}Q}{Q} = \ln\left(\frac{Q(t')}{Q(0)}\right).$$

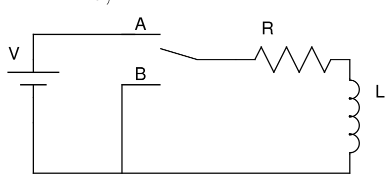
Solving for $Q(t)$, we get,

$$Q(t) = Q(0)e^{-t/RC} = \frac{V^2}{2}e^{-t/RC}.$$

So, at time $T = 2RC$, we get,

$$Q(t = 2RC) = \frac{V^2}{2e^2}.$$

11. At $t = 0$, the switch is closed to terminal A.



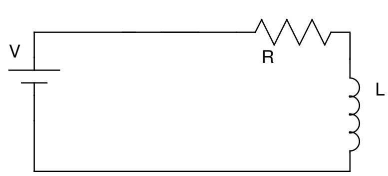
(a) At $t = \tau_1$ how much power is being drawn from the battery? How much of that power is dissipated as heat?

(b) The maximum energy stored in inductor L?

(c) When the current through R is I_0, switch is closed to terminal B. How much time passes before the current falls to I_0/e?

Answer:

(a) When the switch is closed to A, then the circuit looks like:



In order to find the power expended by the battery, we need to find the current in the circuit. And in order to find that, we use Kirchhoff's loop rule:

$$V - IR - L\frac{\mathrm{d}I}{\mathrm{d}t} = 0,$$

so it's a differential equation involving $I(t)$, and as usual, we solve it by separation of variables:

$$\frac{R}{L}\,\mathrm{d}t = \frac{\mathrm{d}I}{V/R - I},$$

and integrating both sides from $t' = 0$ to $t' = t$,

$$\frac{R}{L}\int_0^t \mathrm{d}t = \frac{Rt}{L} = \int_0^t \frac{\mathrm{d}I}{V/R - I} = -\ln\left(\frac{V/R - I(t)}{V/R - I(0)}\right),$$

and solving first for $(V/R - I(t))$,

$$V/R - I(t) = (V/R - I(0))e^{-t/(L/R)},$$

and since $I(0) = 0$, when we solve for $I(t)$, we get,

$$I(t) = \frac{V}{R}\left(1 - e^{-t/(L/R)}\right).$$

So at $t = \tau_1$, the current being drawn from the battery is $I(\tau_1) = \frac{V}{R}\left(1 - e^{-\tau_1/(L/R)}\right)$, so the power drawn from the battery is,

$$P = IV = \frac{V^2}{R}\left(1 - e^{-\tau_1/(L/R)}\right).$$

The power dissipated as heat is the power dissipated by the resistor, which is given by,

$$P_R = I^2R = \frac{V^2}{R}\left(1 - e^{-\tau_1/(L/R)}\right)^2.$$

The rest of the power is stored by the inductor.

(b) The energy stored by the inductor is given by $E = LI^2/$, and the maximum current is $I_{\mathrm{max}} = V/R$, so the maximum energy stored in the inductor is,

$$E_{\mathrm{max}} = \frac{LV^2}{2R^2}.$$

When the switch is closed to B, it is a simple LR circuit and the energy stored in the inductor will be slowly dissipated by the resistor. To solve this, again we use Kirchhoff's loop rule:

$$-L\frac{\mathrm{d}I}{\mathrm{d}t} - IR = 0.$$

Again, solving this differential equation by separation of variables,

$$-\frac{R\,\mathrm{d}t}{L} = \frac{\mathrm{d}I}{I},$$

and we integrate it from $t' = 0$ to $t' = t$:

$$-\frac{R}{L}\int_0^t \mathrm{d}t = -\frac{Rt}{L} = \int_0^t \frac{\mathrm{d}I}{I} = \ln\left(\frac{I(t)}{I(0)}\right),$$

And solving for $I(t)$,

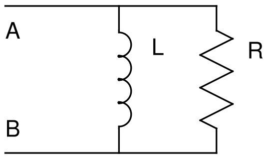
$$I(t) = I_0 e^{-t/(L/R)},$$

So, at time $t = (L/R)$, the current will be,

$$I(t = L/R) = \frac{I_0}{e}.$$

AC circuits

12. Terminals A and B are connected to a signal source.



At 60 Hz, the impedance between A and B is $(300 + 60j)\ \Omega$. Find the inductance L and the resistance R. What is the impedance at 50 Hz?

Answer: To find the impedance at 50 Hz, we have to find the resistance and the inductance.

Impedances add like resistors. So, adding the impedance of the resistor and inductor in parallel, we get,

$$\frac{1}{Z_{\text{eq}}} = \frac{1}{Z_R} + \frac{1}{Z_L},$$

or, solving for Z_{eq} and plugging in the values of impedance for resistor (R) and inductor ($j\omega L$; in electrical engineering, letter j is used for the imaginary number, i.e. $j = \sqrt{-1}$, because i is often used for current),

$$\begin{aligned} Z_{\text{eq}} &= \frac{Z_R Z_L}{Z_R + Z_L} \\ &= \frac{R(j\omega L)}{R + j\omega L}, \end{aligned}$$

to rationalize the fraction, we multiply by "1"

$$= \frac{R(j\omega L)}{R + j\omega L} \cdot \frac{R - j\omega L}{R - j\omega L} = \frac{\omega^2 R L^2 + j\omega R^2 L}{R^2 + \omega^2 L^2}.$$

Equating this with $(300 + 60j)\ \Omega$, we get,

$$\begin{aligned} \frac{\omega^2 R L^2}{R^2 + \omega^2 L^2} &= 300\ \Omega \\ \frac{\omega R^2 L}{R^2 + \omega^2 L^2} &= 60\ \Omega, \end{aligned}$$

So,

$$\frac{300\ \Omega}{60\ \Omega} = \frac{\omega^2 R L^2}{\omega R^2 L} = \frac{\omega L}{R},$$

so,

$$5R = \omega L.$$

We plug this back into the first equation, which gives,

$$\frac{R(5R)^2}{R^2 + (5R)^2} = \frac{25R^3}{26R^2} = \frac{25}{26}R = 300\ \Omega,$$

so,

$$R = 312\ \Omega.$$

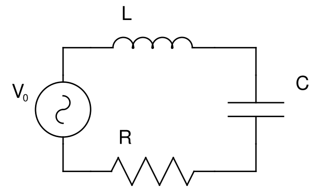
And,

$$L = \frac{5R}{\omega} = \frac{5R}{2\pi \times (60\text{ Hz})} \approx 4.138\text{ H}.$$

For the total impedance at 50 Hz, we use the same formula with $\omega = 2\pi \times 50$ Hz,

$$\begin{aligned} Z_{\text{eq}} &= \frac{\omega^2 R L^2 + j\omega R^2 L}{R^2 + \omega^2 L^2} \\ &= \frac{527280000\ \Omega^3 + j \cdot 126547200\ \Omega^3}{1787344} \\ &\approx (295 + 70.8j)\ \Omega. \end{aligned}$$

13. Consider the below circuit with AC voltage source. $V_0 = 10$ V, peak voltage, $L = 1$ H, $C = 100\ \mu$F, and $R = 50\ \Omega$.



(a) At 60 Hz, how much power is dissipated?

(b) At what frequency is the maximum power dissipated by the resistor, and what is this power?

Answer:

(a) Since impedances add like resistances, the total impedance of the circuit is (note: impedance of a capacitor is given by $1/j\omega C$),

$$\begin{aligned} Z_{\text{eq}} &= Z_L + Z_C + Z_R \\ &= j\omega L + \frac{1}{j\omega C} + R \\ &= R + j\left(\omega L - \frac{1}{\omega C}\right) \end{aligned}$$

So, the magnitude of the impedance is,

$$\begin{aligned} |Z_{\text{eq}}| &= \sqrt{(R + j(\omega L - 1/\omega C))(R - j(\omega L - 1/\omega C))} \\ &= \sqrt{R^2 + (\omega L - 1/\omega C)^2} \approx 354\ \Omega. \end{aligned}$$

From the generalized Ohm's law, peak current is given from the peak voltage,

$$I_p = \frac{V_0}{|Z_{\text{eq}}|} = \frac{10\text{ V}}{354\ \Omega} \approx 28.2\text{ mA},$$

But the average power dissipated by resistor is given by rms (root-mean-square) current, and for sinusoidal wave, the factor between rms current and peak current is $\sqrt{2}$ (comes from the fact $\sqrt{\frac{1}{2\pi}\int_0^{2\pi} \sin^2(t)\ \mathrm{d}t} = \sqrt{1/2}$), i.e.

$$I_{\text{rms}} = \frac{I_p}{\sqrt{2}},$$

and the power dissipated by the resistor is,

$$P = I_{\text{rms}}^2 R = \frac{I_p^2}{2} R = 19.95\text{ mW}.$$

(b) To maximize power dissipation, we have to maximize the current. To do that, we minimize $|Z_{\text{eq}}|$,

$$|Z_{\text{eq}}| = \sqrt{R^2 + (\omega L - 1/\omega C)^2}.$$

There is nothing we can do about the real part of Z_{eq}, i.e. R, because there is nothing to cancel that out with. But if we can somehow make $\omega L - 1/\omega C = 0$, then we can make the imaginary part go away so that $|Z_{\text{eq}}| = R$.

So, let $\omega L = 1/\omega C$, or,

$$\omega^2 = \frac{1}{LC},$$

so that,

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(1\text{ H}) \cdot (10^{-4}\text{ F})}} = 100\text{ s}^{-1},$$

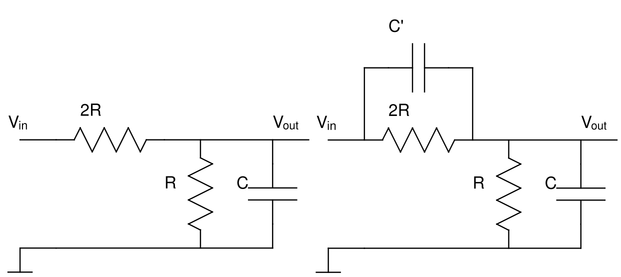
which is equal to $2\pi f$, so the frequency is,

$$f = \frac{100\text{ s}^{-1}}{2\pi} \approx 15.9\text{ Hz}.$$

Since $Z_{\text{eq}} = R$ at this frequency,

$$P = I_{\text{rms}}^2 R = \frac{V^2}{2R^2} R = \frac{V^2}{2R}.$$

14. *from Physics 111-BSC, Lab 2*



(a) In the circuit above on the left, find the frequency at which $V_{\text{out}} = V_{\text{in}}/4$ (in rms).

(b) By adding a capacitor as shown (above on the right), we can make $V_{\text{out}} = V_{\text{in}}/3$ regardless of frequency. Find the correct capacitance C'.

Answer:

(a) If we can simplify R and C into one circuit element with a complex impedance Z_1, then the answer is given by the voltage divider equation (which you can derive by finding the current through the circuit and the voltage drop across Z_1),

$$V_{\text{out}} = \frac{Z_1}{2R + Z_1} V_{\text{in}}.$$

So, we calculate Z_1 by adding the capacitor and the resistor in parallel:

$$\frac{1}{Z_1} = \frac{1}{R} + j\omega C = \frac{1 + j\omega RC}{R}.$$

So,

$$\begin{aligned} V_{\text{out}} &= \frac{R/(1 + j\omega RC)}{2R + R/(1 + j\omega RC)} V_{\text{in}} \\ &= \frac{R}{3R + 2j\omega R^2 C} V_{\text{in}}. \end{aligned}$$

Since we are interested in rms quantities (i.e. ignore phase), we can work with the norms of complex quantities, which gives (by multiplying each side by its own complex conjugate and then taking the square root):

$$V_{\text{out,rms}} = \frac{R}{\sqrt{9R^2 + 4\omega^2 R^4 C^2}} V_{\text{in,rms}} = \frac{1}{4} V_{\text{in,rms}},$$

and we solve for ω $(= 2\pi f)$:

$$\omega = 2\pi f = \sqrt{\frac{7}{4}} \frac{1}{RC}.$$

(b) We can proceed as before, except instead of just using $2R$, we have to compute Z_2 by adding $2R$ and C' in parallel. By same steps as in (a):

$$Z_2 = \frac{2R}{1 + 2j\omega RC'}.$$

And the voltage divider equation gives,

$$V_{\text{out}} = \frac{Z_1}{Z_1 + Z_2} V_{\text{in}}.$$

Doing this directly could involve some complicated algebra to find the value of C' necessary to cancel ω out of the expression. So we try a simpler method. First, we simplify the equation above to obtain something simpler:

$$V_{\text{out}} = \frac{1}{1 + Z_2/Z_1} V_{\text{in}},$$

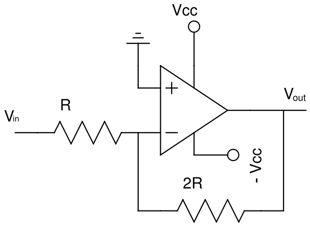
and we see that if we could eliminate ω in the expression Z_2/Z_1, we are done. The ratio is:

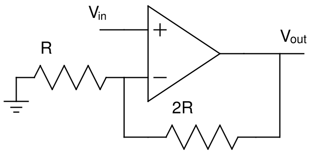
$$\frac{Z_2}{Z_1} = \frac{2R/(1 + 2j\omega RC')}{R/(1 + j\omega RC)} = 2\frac{1 + j\omega RC}{1 + 2j\omega RC'}.$$

By inspection, we see that if $C' = C/2$, the fraction will cancel to 1 one we will have $Z_2/Z_1 = 2$, giving us $V_{\text{out}} = V_{\text{in}}/3$.

Op Amp circuits

15. *Amplifiers*





(a) For the circuit above on the left ("inverting amplifier"), give V_{out} in terms of V_{in}.
(b) For the circuit above on the right ("non-inverting amplifier"), give V_{out} in terms of V_{in}.

Answer:

(a) We start by identifying the voltage at the two inputs. The noninverting input is at 0 V, because it's grounded. Then, by the first golden rule, the inverting input has to be at 0 V.

This gives the voltage difference across the resistor R, so we can calculate the current through it:

$$I = \frac{\Delta V}{R} = \frac{V_{\text{in}} - 0}{R}.$$

Now, by the second golden rule, none of this current goes into the noninverting input, so it has to go through the resistor $2R$. This gives the voltage difference between noninverting input and V_{out},

$$\Delta V = 0 - V_{\text{out}} = I(2R) = 2V_{\text{in}},$$

so,

$$V_{\text{out}} = -2V_{\text{in}}.$$

(b) We go through the same steps. First, identify the voltage at both inputs using the first golden rule. The noninverting input is at V_{in}, so the inverting input has to be at V_{in} as well. This gives the voltage difference across the resistor R, so we can calculate the current through R:

$$I = \frac{\Delta V}{R} = \frac{0 - V_{\text{in}}}{R}.$$

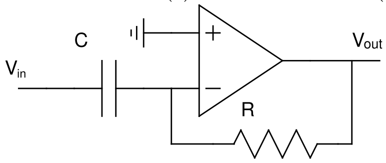
Again, by the second golden rule, all this current has to go through the $2R$ resistor, which gives the voltage difference between the inverting input and V_{out}:

$$\Delta V = V_{\text{in}} - V_{\text{out}} = I(2R) = -\frac{V_{\text{in}}}{R}(2R),$$

so, solving for V_{out},

$$V_{\text{out}} = 3V_{\text{in}}.$$

16. (a) Find the $V_{\text{out}}(t)$ in terms of $V_{\text{in}}(t)$, R, and C.



(b) Analyze the circuit in (a) with the positions of capacitor and resistor swapped.

Answer:

(a) By the first golden rule, the voltage at the inverting input is equal to the voltage at the noninverting input, 0. This gives the voltage difference across C, which gives the charge on the capacitor:

$$Q = C\Delta V = C(V_{\text{in}} - 0).$$

The current through the capacitor is given by $I = \frac{\mathrm{d}Q}{\mathrm{d}t}$, so we differentiate the expression above to get the current,

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t} = C\frac{\mathrm{d}V_{\text{in}}}{\mathrm{d}t}.$$

By the second golden rule, this goes through R, which gives the voltage difference across the resistor,

$$\Delta V = 0 - V_{\text{out}} = IR = CR\frac{\mathrm{d}V_{\text{in}}}{\mathrm{d}t},$$

so the output voltage is given by,

$$V_{\text{out}} = -CR\frac{\mathrm{d}V_{\text{in}}}{\mathrm{d}t}.$$

(b) Using the first golden rule, the voltage at the inverting input is equal to the voltage at the noninverting input, 0. This gives the voltage difference across the resistor R, hence the current through R:

$$I = \frac{\Delta V}{R} = \frac{V_{\text{in}} - 0}{R}.$$

By the second golden rule, all this current has to go through the capacitor. Since the current through the capacitor is related to the charge on the capacitor by $I = \frac{\mathrm{d}Q}{\mathrm{d}t}$, we get,

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = I = \frac{V_{\text{in}}}{R},$$

so the charge on the capacitor is given by,

$$Q(t) = \int_0^t \frac{V_{\text{in}}(t)}{R}\ \mathrm{d}t,$$

which in turn, gives the voltage drop across the capacitor,

$$\Delta V = 0 - V_{\text{out}} = \frac{Q}{C} = \frac{1}{RC}\int_0^t V_{\text{in}}(t)\ \mathrm{d}t,$$

so the output voltage is given by,

$$V_{\text{out}} = -\frac{1}{RC}\int_0^t V_{\text{in}}(t)\ \mathrm{d}t.$$